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The asymptotic iteration method for the eigenenergies of the Schrödinger equation with the potential $V(r) = -Z/r + gr + \lambda r^2$

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Abstract

The eigenenergies of the Schrödinger equation with Coulomb potential plus linear and harmonic radial terms are investigated using the perturbation technique within the framework of the asymptotic iteration method. The obtained eigenenergies are compared with those obtained from the shifted $1/N$ expansion method, the moment method and also with the exact values obtained from the super-symmetric method. It is found that asymptotically this method gives accurate results for arbitrary g, Z and λ.

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1. Introduction

Since the appearance of the Schrödinger equation in quantum mechanics, there has been continual research into solving the Schrödinger equation with exactly solvable potentials using different methods and techniques [1–8].

In recent years, the study of the Coulomb potential perturbed by a term or terms involving various powers of r has become much more desirable for the theoretical understanding of several newly discovered phenomena in different branches of physics. The lack of an exact analytical solution to such potentials makes the study of these potentials one of the most popular theoretical laboratories for examining the validity of various approximation techniques based on perturbative and nonperturbative approaches.

A well-known and important example in this respect is the Coulomb potential plus linear and harmonic radial terms

$$V(r) = \frac{-Z}{r} + gr + \lambda r^2. \tag{1}$$

This potential has played an important role in many different fields of physics, such as atomic and molecular physics [9, 10], particle physics [11–16], plasma physics and solid-state physics [17–19].

This potential has been studied in the context of the Rayleigh–Schrödinger perturbation method [10, 20], the moment method [21], analytic continued fraction theory [22–24], the Hill determinant [25, 26], the Bender–Wu method [17], the Padé method [9], super-symmetric quantum mechanics (SUSYQM) [27], the shifted $1/N$ expansion method [28], the two-point quasi-fractional approximations method [29] and other methods [30, 31].

However, explicit calculation of the eigenenergies with the Rayleigh–Schrödinger perturbation theory, which is described in most quantum mechanics textbooks, runs into difficulty with the summation over all intermediate unperturbed eigenstates. In addition, the logarithmic perturbation theory (LPT) and the Dalgarno–Lewis technique [32] all have problems in calculating corrections to the excited states. Thus, the need arises to have a relatively simple and effective method which will give, to a high degree of accuracy, the eigenenergies ($E_{n\ell}$) and eigenfunctions without any constraint on the potential parameter values involved.

In recent years, much attention has been focused on the asymptotic iteration method (AIM) [33]. This method reproduces exact solutions to many differential equations which are important in applications to many problems in physics, such as the equations of Hermite, Laguerre, Legendre and Bessel [33]. The AIM also gives complete exact solutions of the Schrödinger equation for the Pösch–Teller potential, the harmonic oscillator potential, the complex cubic potential [34] and the anharmonic oscillator potentials [35, 36]. Very recently we applied the AIM and we found exact eigenenergies for the angular spheroidal wave equation [37, 38].

Encouraged by its satisfactory performance through comparisons with the other methods, and the importance of the Coulomb potential plus linear and harmonic radial terms, we feel tempted to extend the AIM to solve the three-dimensional Schrödinger equation for this potential. However, this time not by solving the equation directly, as we have done earlier [36], rather by using an alternative approach to the perturbation theory within the framework of the AIM [39]. Shortly, we shall see how the AIM can be used to find the coefficients in the perturbation series for the three-dimensional Schrödinger equation directly, without either using the base eigenfunctions of the unperturbed problem or needing to calculate matrix elements. This method is applicable in the same form to both the ground state and excited states without involving tedious calculations which appear in the available perturbation theories.

With this in mind, this paper is organized as follows. In section 2, the formulation of the AIM for finding the eigenenergies of a Coulomb potential in the presence of external linear plus harmonic oscillator terms is outlined. The analytical expressions for AIM are cast in such a way that allows the reader to use them without proceeding into their derivation. In section 3, we explain the method to obtain numerically the eigenenergies, and therein we will compare the accuracy of our results with those obtained by other methods, such as the shifted $1/N$ expansion method, the moment method and the super-symmetric method. Finally, we remark on the results and our findings.

2. Formalism of the asymptotic iteration method for the potential $V(r) = -Z/r + gr + \lambda r^2$

Consider the three-dimensional radial Schrödinger equation (in atomic units, $\hbar = m = c = 1$)

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\ell(\ell+1)}{2r^2} - \frac{Z}{r} + gr + \lambda r^2\right]\Psi_{n\ell}(r) = E_{n\ell}\Psi_{n\ell}(r), \tag{2}$$

with $\Psi_{n\ell}(0) = 0$ and $\Psi_{n\ell}(\infty) = 0$. $E_{n\ell}$s are the eigenenergies, Z represents the variable (positive) nuclear charge, g and λ represent independent coupling constants.

Making the change of variables $r = u^2$, and furthermore we remove the first derivative by proposing the ansatz

$$\Phi_{n\ell}(u) = \Psi_{n\ell}(u)\exp(-p(u)/2); \qquad p'(u) = -\frac{1}{u}, \tag{3}$$

which in turn implies a Schrödinger-like radial equation

$$\frac{\mathrm{d}^2\Phi_{n\ell}(u)}{\mathrm{d}u^2} - \left(\frac{\acute{\ell}(\acute{\ell}+1)}{u^2} - 8E_{n\ell}u^2 + 8gu^4 + 8\lambda u^6 - 8Z\right)\Phi_{n\ell}(u) = 0, \tag{4}$$

with $\acute{\ell}(\acute{\ell}+1) = 4\ell(\ell+1) + \frac{3}{4}$.

If we further insert the ansatz

$$\Phi_{n\ell}(u) = u^{(\acute{\ell}+1)}\mathrm{e}^{-\alpha u^4/4} f_{n\ell}(u) \tag{5}$$

into equation (4), carry out the mathematics, collect the terms of equal powers in u, and choosing the arbitrary parameter $\alpha = \sqrt{8\lambda}$, then the function $f_{n\ell}(u)$ will satisfy a new second-order homogeneous linear differential equation of the form

$$f''_{n\ell}(u) = \lambda_0(u,\gamma) f'_{n\ell}(u) + s_0(u,\gamma) f_{n\ell}(u), \tag{6}$$

where

$$\lambda_0(u,\gamma) = 2\left(\alpha u^3 - \frac{(\acute{\ell}+1)}{u}\right), \qquad s_0(u,\gamma) = (-\epsilon_{n\ell}u^2 + \gamma(8gu^4 - 8Z)), \tag{7}$$

$$\epsilon_{n\ell} = 8E_{n\ell} - (2\acute{\ell}+5)\alpha, \tag{8}$$

and γ is an artificially introduced perturbation expansion parameter to be equal to 1 at the end of the calculations.

Here, it should be pointed out that the choice of the ansatz in equation (5) is motivated, that is when we switch off γ, equation (6) will be reduced to an exactly solvable eigenvalue problem within the framework of AIM [34].

However, straightforward application of AIM to equation (6), when $\gamma \neq 0$, does not give any reasonable results. Where it is observed that a different number of iterations are required to obtain different eigenstates and the sequence appears to converge when the number of iterations $k \leqslant \simeq 30$, but then starts to oscillate as k increases. This result violates the principle behind the AIM.

Therefore, to overcome this problem, we will use a perturbation approach within the framework of the AIM. Hence, we propose to use in equation (6) a potential $V(u)$, which is expressed as a sum of two parts

$$V(u) = V_0(u) + \gamma V_1(u), \tag{9}$$

with $V_0(u) = -\epsilon_{n\ell}u^2$ to have an exact solution within AIM and $V_1(u) = \gamma(8gu^4 - 8Z)$ which will be investigated using a perturbation expansion technique within the framework of AIM.

Following the systematic procedure of AIM [33], that will lead to a general solution of the second-order differential equation (6), we get

$$f''_{n\ell}(u) = \exp\left(-\int^u \varrho(u',\gamma)\,\mathrm{d}u'\right)\left[C_2 + C_1\int^u \exp\left(\int^{u'}\{\lambda_0(u'',\gamma) + 2\varrho(u'',\gamma)\}\,\mathrm{d}u''\right)\mathrm{d}u'\right], \tag{10}$$

if for some $k > 0$,

$$\varrho(u,\gamma) \equiv \frac{s_k(u,\gamma)}{\lambda_k(u,\gamma)} = \frac{s_{k-1}(u,\gamma)}{\lambda_{k-1}(u,\gamma)}, \tag{11}$$

with

$$\lambda_k(u,\gamma) = \lambda'_{k-1}(u,\gamma) + s_{k-1}(u,\gamma) + \lambda_0(u,\gamma)\lambda_{k-1}(u,\gamma), \tag{12}$$

and

$$s_k(u,\gamma) = s'_{k-1}(u,\gamma) + s_0(u,\gamma)\lambda_{k-1}(u,\gamma). \tag{13}$$

Here, it should be noted that one can construct the eigenfunctions $f_{n\ell}(u)$ from the knowledge of $\varrho(u,\gamma)$. For sufficiently large k, one can also introduce the termination condition of the method

$$\delta_k(u,\gamma) \equiv s_k(u,\gamma)\lambda_{k+1}(u,\gamma) - s_{k+1}(u,\gamma)\lambda_k(u,\gamma) = 0. \tag{14}$$

If we expand $\delta_k(u,\gamma)$ around $\gamma = 0$, we get the following series:

$$\delta_k(u,\gamma) = \delta_k(u,0) + \frac{\gamma}{1!}\frac{\partial\delta_k(u,\gamma)}{\partial\gamma}\bigg|_{\gamma=0} + \frac{\gamma^2}{2!}\frac{\partial^2\delta_k(u,\gamma)}{\partial\gamma^2}\bigg|_{\gamma=0} + \frac{\gamma^3}{3!}\frac{\partial^3\delta_k(u,\gamma)}{\partial\gamma^3}\bigg|_{\gamma=0} + \cdots. \tag{15}$$

According to the procedure of AIM, $\delta_k(u,\gamma)$ must be zero; if this to be true for every γ value, then every term of the series must be zero. That is to say

$$\delta_k^{(j)}(u,\gamma) = \frac{\gamma^j}{j!}\frac{\partial^j\delta_k(u,\gamma)}{\partial\gamma^j}\bigg|_{\gamma=0} = 0, \qquad j = 0, 1, 2, \ldots. \tag{16}$$

Up to this point, it is also convenient to expand $\epsilon_{n\ell}$,

$$\epsilon_{n\ell} = \epsilon_{n\ell}^{(0)} + \gamma\epsilon_{n\ell}^{(1)} + \gamma^2\epsilon_{n\ell}^{(2)} + \gamma^3\epsilon_{n\ell}^{(3)} + \gamma^4\epsilon_{n\ell}^{(4)} + \cdots. \tag{17}$$

A quantitative estimate for $\epsilon_{n\ell}$ expansion terms can be obtained by comparing the terms with the same order of γ in equations (16) and (17). Therefore, it is clear that the roots of $\delta_k^{(0)}(u,0) = 0$ give us the leading contribution energy terms $\epsilon_{n\ell}^{(0)}$. Likewise, the roots of $\delta_k^{(1)}(u,\gamma) = 0$ give us the first correction terms to $\epsilon_{n\ell}^{(1)}$ and so on. Accordingly, the general solution for the eigenenergies $E_{n\ell}$ in conjunction with equation (8) is

$$E_{n\ell} = \tfrac{1}{8}\big((2\acute{\ell}+5)\alpha + \epsilon_{n\ell}^{(0)} + \gamma\epsilon_{n\ell}^{(1)} + \gamma^2\epsilon_{n\ell}^{(2)} + \gamma^3\epsilon_{n\ell}^{(3)} + \gamma^4\epsilon_{n\ell}^{(4)} + \cdots\big). \tag{18}$$

3. Numerical results for the eigenenergies of the potential $V(r) = -Z/r + gr + \lambda r^2$

Within the framework of the asymptotic iteration method mentioned in the above section, the eigenenergies $E_{n\ell}$ of the Coulomb potential in the presence of external linear plus harmonic oscillator radial terms are calculated by means of equation (18).

To obtain the leading energy term $\epsilon_{n\ell}^{(0)}$, one should simply switch off γ in equation (6), which will lead to an exactly solvable eigenvalue problem within the framework of AIM [34]

$$f''_{n\ell}(u) = 2\left(\alpha u^3 - \frac{(\acute{\ell}+1)}{u}\right) f'_{n\ell}(u) - \epsilon_{n\ell}u^2 f_{n\ell}(u), \tag{19}$$

the roots of $\delta_k^{(0)}(u,0) = 0$ yield

$$\epsilon_{n\ell}^{(0)} = 8n\alpha, \qquad n = 0, 1, 2, \ldots. \tag{20}$$

For each iteration, the expression $\delta_k^{(0)}(u,0) = 0$ depends on two variables, namely, $\epsilon_{n\ell}^{(0)}$ and u. Since the problem is exactly solvable, the calculated eigenenergies $\epsilon_{n\ell}^{(0)}$ by means of this condition are independent of the choice of u. Nevertheless, the choice of u is observed to be critical to the speed of the convergence. Therefore, for this problem we choose

Table 1. The calculated values of the coefficients in the energy expansion and the total energy $\epsilon_{n\ell}$ by means of this work for different values of ℓ, Z, λ and g.

n	ℓ	Z	λ	g	$\epsilon_{n\ell}^{(0)}$	$\epsilon_{n\ell}^{(1)}$	$\epsilon_{n\ell}^{(2)}$	$\epsilon_{n\ell}^{(3)}$	$\epsilon_{n\ell}$
0	0	1	0.1	0.447 21	0	0	−4.000 00	0	−4.000 00
0	1	1	0.1	0.223 61	0	0	−1.000 00	0	−1.000 00
0	2	1	0.1	0.149 07	0	0	−0.444 44	0	−0.444 44
0	3	1	0.1	0.111 80	0	0	−0.250 00	0	−0.250 00
0	0	1	10.0	4.472 14	0	0	−4.000 00	0	−4.000 00
0	1	1	10.0	2.236 07	0	0	−1.000 00	0	−1.000 00
0	2	1	10.0	1.490 71	0	0	−0.444 44	0	−0.444 44
0	3	1	10.0	1.118 03	0	0	−0.250 00	0	−0.250 00
0	0	10	0.1	4.472 14	0	0	−400.000	0	−400.000
0	1	10	0.1	2.236 07	0	0	−100.000	0	−100.000
0	2	10	0.1	1.490 71	0	0	−44.444 44	0	−44.444 44
0	3	10	0.1	1.118 03	0	0	−25.0000	0	−25.0000
0	0	10	10.0	44.721 36	0	0	−400.000	0	−400.000
0	1	10	10.0	22.360 68	0	0	−100.000	0	−100.000
0	2	10	10.0	14.907 12	0	0	−44.444 44	0	−44.444 44
0	3	10	10.0	11.180 34	0	0	−25.0000	0	−25.0000
0	0	10	1000.0	447.213 59	0	0	−400.000	0	−400.000
0	1	10	1000.0	223.606 80	0	0	−100.000	0	−100.000
0	2	10	1000.0	149.071 20	0	0	−44.444 44	0	−44.444 44
0	3	10	1000.0	111.803 40	0	0	−25.0000	0	−25.0000

Table 2. The calculated values of the coefficients in the energy expansion and the total energy $\epsilon_{n\ell}$ by means of this work for $Z = 1$, $\lambda = 1$ and for different values of g.

g	$\epsilon_{00}^{(0)}$	$\epsilon_{00}^{(1)}$	$\epsilon_{00}^{(2)}$	$\epsilon_{00}^{(3)}$	$\epsilon_{00}^{(4)}$	$\epsilon_{00}^{(5)}$	$\epsilon_{n\ell}$
−2.0	0	−25.916 60	−0.337 33	−0.065 79	−0.016 72	−0.005 01	−26.341 45
−1.0	0	−18.325 81	−0.354 58	−0.078 95	−0.016 94	−0.003 22	−18.780 00
−0.5	0	−14.530 41	−0.691 11	−0.151 01	−0.027 01	−0.003 19	−15.402 73
−0.1	0	−11.494 10	−1.117 74	−0.204 11	−0.027 26	−0.000 81	−12.843 29
0.0	0	−10.735 00	−1.246 25	−0.214 11	−0.025 99	−0.000 03	−12.221 38
0.1	0	−9.975 93	−1.383 51	−0.222 18	−0.024 23	−0.000 76	−11.606 61
0.5	0	−6.939 61	−2.020 00	−0.228 75	−0.013 42	0.003 43	−9.198 35
1.0	0	−3.144 21	−3.012 35	−0.155 48	0.000 89	0.003 55	−6.307 60
2.0	0	4.446 59	−5.632 87	0.412 74	−0.056 04	−0.000 64	−0.855 94

$u = \left(\frac{\ell+1}{\alpha}\right)^{(1/4)}$ which corresponds to the maximum point of the asymptotic wavefunction given in equation (5), and the exact eigenenergies begin to appear when the number of iterations $k \geqslant 25$.

On the other hand, to obtain the perturbative expansion terms, first one should switch on γ, substitute $\epsilon_{n\ell}^{(0)} + \gamma\epsilon_{n\ell}^{(1)}$ for $\epsilon_{n\ell}$ in equation (6) and terminate the iterations by imposing the condition $\delta_k^{(1)}(u, \gamma) = 0$ as an approximation to equation (6). The first root of the resulting equation gives $\epsilon_{n\ell}^{(1)}$. Similarly, one can obtain the other perturbative expansion terms.

The results of these latter calculations do not lend themselves to symbolic expression. Especially $\epsilon_{n\ell}^{(1)}$, $\epsilon_{n\ell}^{(2)}$, $\epsilon_{n\ell}^{(3)}$, $\epsilon_{n\ell}^{(4)}$ and $\epsilon_{n\ell}^{(5)}$ tend to become rather numerical; consequently, in tables 1–3, an explicit list of calculations with different values of n, ℓ, Z, λ and g are given, so that the reader may, if so inclined, reproduce our results.

Table 3. The calculated values of the coefficients in the energy expansion and the total energy $\epsilon_{n\ell}$ by means of this work for different values of ℓ, Z, λ and g.

n	ℓ	Z	λ	g	$\epsilon_{n\ell}^{(0)}$	$\epsilon_{n\ell}^{(1)}$	$\epsilon_{n\ell}^{(2)}$	$\epsilon_{n\ell}^{(3)}$	$\epsilon_{n\ell}^{(4)}$	$\epsilon_{n\ell}$
1	0	1	0.1	0.096 20	$\frac{16}{\sqrt{5}}$	−3.082 77	−0.723 13	0.014 26	0.035 40	3.399 18
0				0.574 62	0	1.719 80	−5.104 59	0.456 02	−0.090 75	−3.019 52
1	1	1	0.1	0.009 69	$\frac{16}{\sqrt{5}}$	−3.395 32	−0.177 29	−0.007 86	0.000 00	3.574 95
0				0.362 98	0	2.508 45	−1.679 28	0.127 05	−0.013 16	0.943 06
1	2	1	0.1	−0.020 25	$\frac{16}{\sqrt{5}}$	−3.520 70	−0.062 30	−0.002 79	0.000 00	3.569 63
0				0.281 13	0	2.852 17	−0.905 68	0.054 59	−0.003 94	1.997 14
1	3	1	0.1	−0.033 59	$\frac{16}{\sqrt{5}}$	−3.573 63	−0.025 57	−0.001 01	0.000 00	3.555 21
0				0.234 84	0	3.036 92	−0.589 58	0.029 08	0.000 00	2.476 42
1	0	1	10.0	−3.426 12	$32\sqrt{5}$	−37.845 40	−0.269 89	−0.010 25	0.000 00	33.428 64
0				10.134 32	0	24.169 70	−9.995 30	1.249 50	−0.205 85	15.218 05
1	1	1	10.0	−3.609 56	$32\sqrt{5}$	−38.160 80	−0.211 52	−0.010 02	0.000 000	33.171 84
0				7.336 34	0	29.028 20	−4.359 20	0.378 95	−0.037 35	25.010 60
1	2	1	10.0	−3.428 01	$32\sqrt{5}$	−37.883 60	−0.233 15	−0.010 14	0.000 00	33.427 29
0		1		6.036 75	0	31.048 50	−2.725 60	0.177 71	−0.012 56	28.488 05
1	3	1	10.0	−3.224 73	$32\sqrt{5}$	−37.607 60	−0.239 59	−0.009 26	0.000 00	33.697 73
0		1		5.237 19	0	32.152 00	−1.956 80	0.101 64	0.000 00	30.296 84

Table 4. Comparison between selected eigenenergies calculated from the shifted $(1/N)$ expansion method $E(1/N)$, the exact super-symmetric values E(SUSY) and the eigenenergies $E_{n\ell}$(AIM) computed by means of equation (18).

n	ℓ	Z	λ	g	$E(1/N)$	E(SUSY)	E(AIM)
0	0	1	0.1	0.447 21	0.171 66	0.170 82	0.170 82
0	1	1	0.1	0.223 61	0.993 37	0.993 03	0.993 03
0	2	1	0.1	0.149 07	1.509 79	1.509 69	1.509 69
0	3	1	0.1	0.111 80	1.981 24	1.981 21	1.981 21
0	0	1	10.0	4.472 14	6.226 80	6.208 20	6.208 20
0	1	1	10.0	2.236 07	11.057 19	11.055 34	11.055 34
0	2	1	10.0	1.490 71	15.597 32	15.596 92	15.596 92
0	3	1	10.0	1.118 03	20.093 49	20.093 36	20.093 36
0	0	10	0.1	4.472 14	−49.329 18	−49.329 18	−49.329 18
0	1	10	0.1	2.236 07	−11.381 96	−11.381 97	−11.381 97
0	2	10	0.1	1.490 71	−3.990 30	−3.990 31	−3.990 31
0	3	10	0.1	1.118 03	−1.112 53	−1.112 54	−1.112 54
0	0	10	10.0	44.721 36	−43.291 58	−43.291 80	−43.291 80
0	1	10	10.0	22.360 68	−1.319 03	−1.319 66	−1.319 66
0	2	10	10.0	14.907 12	10.097 63	10.096 92	10.096 92
0	3	10	10.0	11.180 34	17.000 07	16.999 61	16.999 61
0	0	10	1000.0	447.213 59	17.166 39	17.082 04	17.082 04
0	1	10	1000.0	223.606 80	99.336 72	99.303 40	99.303 40
0	2	10	1000.0	149.071 20	150.978 66	150.969 20	150.969 20
0	3	10	1000.0	111.803 40	198.124 48	198.121 12	198.121 12

Table 5. Comparison between selected eigenenergies calculated from the Shifted $E(1/N)$ expansion method, the moment method E (Bessis *et al*) and the eigenenergies $E_{n\ell}$(AIM) computed by means of equation (18).

n	ℓ	Z	λ	g	$E(1/N)$	E (Bessis *et al*)	E(AIM)
0	0	1	1	−2.0	−1.149 33	−1.1717	−1.1715
0				−1.0	−0.218 57	−0.226 19	−0.226 13
0				−0.5	0.202 84	0.196 00	0.196 00
0				−0.1	0.522 47	0.515 94	0.515 82
0	0	1	1	0.0	0.600 25	0.593 77	0.593 65
0				0.1	0.677 25	0.670 81	0.670 50
0				0.5	0.977 90	0.971 62	0.971 52
0				1.0	1.3391	1.332 80	1.332 87
0				2.0	2.0213	2.014 90	2.014 33

Table 6. Comparison between selected eigenenergies calculated from the shifted $E(1/N)$ expansion method, the exact super-symmetric values E (SUSY) and the eigenenergies $E_{n\ell}$(AIM) computed by means of equation (18).

n	ℓ	Z	λ	g	$E(1/N)$	E(SUSY)	E(AIM)
1	0	1	0.1	0.096 20	1.158 40	1.094 90	1.095 72
0				0.574 62	0.293 49	0.292 56	0.293 32
1	1	1	0.1	0.009 69	1.579 47	1.565 01	1.564 90
0				0.362 98	1.236 19	1.235 86	1.235 91
1	2	1	0.1	−0.020 25	2.016 32	2.011 44	2.011 45
0				0.281 13	1 .814 97	1.814 88	1.814 89
1	3	1	0.1	−0.033 59	2.458 93	2.456 85	2.456 86
0				0.234 84	2.321 84	2.321 80	2.322 01
1	0	1	10.0	−3.426 12	11.129 19	10.886 88	10.886 78
0				10.134 32	8.631 16	8.612 73	8.612 19
1	1	1	10.0	−3.609 56	15.379 90	15.326 75	15.326 82
0				7.336 34	14.308 78	14.306 93	14.306 67
1	2	1	10.0	−3.428 01	19.848 92	19.830 83	19.830 89
0		1		6.036 75	19.213 96	19.213 55	19.213 48
1	3	1	10.0	−3.224 73	24.344 50	24.336 77	24.336 83
0		1		5.237 19	23.911 18	23.911 04	23.911 72

Moreover, in tables 4–6 the results of AIM, together with the shifted expansion method $E(1/N)$, the moment method E(Bessis *et al*) and the super-symmetric exact results E(SUSY), are displayed for comparison purposes.

In the tables, we have only considered the eigenenergies for the ground state $n = 0$ and the first excited state $n = 1$ with different values of ℓ. This was in order to make a clear comparison between the results of this method and the results of [21, 28]. An examination of tables 4–6 shows that the accuracy of the AIM is better than the accuracy of shifted $1/N$ expansion method, and the predicted eigenenergies $E_{n\ell}$(AIM) are all in excellent agreement with the results of the moment method [21] and the super-symmetric method [28]. Furthermore, the special case $Z = 0$, $g = 0$ and $\lambda = 1$, which corresponds to the three-dimensional harmonic oscillator, and the cases studied in [40] can be easily reproduced within the framework of this work.

It is also important to emphasize that the approach described in this paper and its analytical expressions are much more useful than pure numerical calculations, since they represented useful aids in modelling realistic physical problems and they offered an interesting field of investigation in their own right.

Finally, although in this work we have focused only on the calculations of the eigenenergies for the potential (1), one can also find analytical and numerical solutions easily for the corresponding eigenfunctions, if necessary, through the use of equations (10) and (11).

As a concluding remark, we have shown that it is very easy task to implement the perturbation technique within the framework of the AIM without having to worry about the ranges of the couplings in the potential. Moreover, the degree of precision of the results can be drastically improved by raising the perturbative order in the expansion a step, which does not bear any technical difficulty.

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